Abelian Varieties II

Luozi Shi

Main reference: David Mumford's Abelian Varieties Chapter 6 and Brian Conrad's notes on abelian varieties.
Convention: Fix an algebraically closed field k and a variety is always assumed to be integral.

0 Preparation

(will copy this part on the blackboard before the seminar starts)
Recall the following results from Talk 11.

Definition 0.1. *An abelian variety X is a proper, smooth and connected group scheme.*

We defined $m : X \times X \to X$, $T_x : X \to X$, $[n] : X \to X$, and so on..

Corollary 0.2. *Let X be an abelian variety, then $H^0(X, \mathcal{O}_X) = k$.*

Theorem 0.3 (Seesaw Theorem). *Let X be a complete variety, T any variety and $\mathcal{L}$ a line bundle on $X \times T$. Then the set*

$$T_1 = \{t \in T(k) \mid \mathcal{L}|_{X\times\{t\}} \text{ is trivial on } X \times \{t\}\}$$

is closed in $T(k)$, and if on $p_2 : X \times T_1 \to T_1$ is the projection, then $\mathcal{L}|_{X\times T_1} \cong p_2^*\mathcal{M}$ for some line bundle M on T_1.

Corollary 0.4 (Theorem of the square). *For all line bundles $\mathcal{L}$, x, $y \in X(k)$*

$$T_{x+y}^*\mathcal{L} \otimes \mathcal{L} \cong T_x^*\mathcal{L} \otimes T_y^*\mathcal{L}.$$

Proposition 0.5. *Over X the group of Weil divisors are isomorphic to the group of Cartier divisors. Furthermore, the monoid of effective divisors are the isomorphic to the monoid of effective Cartier divisors.*

Proof. Details can be found in Hartshorne Proposition 2.6.11. (requires X to be an integral, separated Noetherian scheme, all of whose local rings are factorial) □

This proposition allows us to use effective divisors and effective Cartier divisors interchangeably.
By the fact $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \iff D_1 \equiv D_2$, we get Thm of the square in divisor form: let D be any divisor (notice that for any divisor D, $\mathcal{O}_X(D)$ is a line bundle now!) then

$$T_{x+y}^*D + D \equiv T_x^*D + T_y^*D$$

where the $\equiv$ sign means they differ by a principal generator.
Define the map

$$\varphi_{\mathcal{L}}(x) = \textit{isom. class of } T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

By theorem of the square $\varphi_{\mathcal{L}}(x)$ is a group homomorphism from $X(k)$ to $\mathrm{Pic}(X)$. Moreover, one can easily check that

(i) $\varphi_{\mathcal{L}_1\otimes\mathcal{L}_2} = \varphi_{\mathcal{L}_1} + \varphi_{\mathcal{L}_2}$

(ii) $\varphi_{T_x^*\mathcal{L}} = \varphi_{\mathcal{L}}$.

1 Characterizing ample line bundles on X

Definition 1.1. *Let X be an abelian variety, $\mathcal{L}$ a line bundle over X then define*

$$K(\mathcal{L}) := \ker(\varphi_{\mathcal{L}}) = \{x \in X(k) \mid T_x^*\mathcal{L} \cong \mathcal{L}\}.$$

Futhermore, if $\mathcal{L} = \mathcal{O}_X(D)$ where D is a Weil divisor, define

$$H(D) := \{x \in X(k) \mid T_x^*(D) = D\}.$$

Again by the fact $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2) \iff D_1 \equiv D_2$ one can see that $H(D) \subset K(L)$.

Proposition 1.2. *$K(\mathcal{L})$ and $H(D)$ are Zariski-closed subgroups of X.*

Proof. $\underline{K(\mathcal{L})}$: Define $(x, \mathrm{id}) : X \to X \times X$ to be the morphism that sends $y \mapsto (x, y)$. It follows that $m \circ (x, \mathrm{id}) = T_x$ and $p_2 \circ (x, \mathrm{id}) = \mathrm{id}_X$ by checking on k-valued points. Consider the line bundle $m^*\mathcal{L} \otimes p_2^*\mathcal{L}^{-1}$ on $X \times X$, one has

$$m^*\mathcal{L} \otimes p_2^*\mathcal{L}^{-1}|_{\{x\}\times X} \cong (x, \mathrm{id})^*(m^*\mathcal{L} \otimes p_2^*\mathcal{L}^{-1}) \cong T_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

the set of x satisfying this equation is exactly $K(\mathcal{L})$. By Seesaw Theorem $K(\mathcal{L})$ is Zariski-closed.

$\underline{H(D)}$: $H(D) = \{x \in X(k) \mid T_x^*(D) = D\} = \bigcap_{d \in D(k)}(-d + D)$ is a closed subgroup. □

Remark. *Here $K(\mathcal{L})/H(D)$ being Zarkski-closed means on the subspace topology on $X(k)$.*

Here comes the main theorem for the characterization of an ample line bundle.

Theorem 1.3. *Let D be an effective divisor (or an effective Cartier divisor) on an abelian variety X and $\mathcal{L} = \mathcal{O}_X(D)$ be the associated line bundle. TFAE:*

(i) $H(D)$ is finite.

(ii) $K(\mathcal{L})$ is finite.

(iii) $\mathcal{L}^{\otimes 2}$ is finitely globally generated and defines a finite morphism $\varphi : X \to \mathbb{P}^N$.

(iv) $\mathcal{L}$ is ample on X.

Proof. *(iii)* $\implies$ *(iv)*: As $\varphi : X \to \mathbb{P}^N$ is induced by global sections, we have $\varphi^*\mathcal{O}(1) \cong L^{\otimes 2}$. It follows by the fact that pullback of ample line bundle under an affine morphism is ample.

(iv) $\implies$ *(ii)*: Let Y be the connected component of 0 in $K(\mathcal{L})$, equipped with the reduced closed subscheme structure.

Claim: Y is an abelian variety.

Proof of the Claim: Group scheme: Consider $K(\mathcal{L})(k)$ as a topological group, and the connected component of the identity of a topological group is a closed subgroup of $X(k)$. As Y is geometrically integral, $Y \times Y$ is integral. And it suffices to check that $m : Y \times Y \to X$ factors through Y.

Proper: Y admits a close immersion into X.

Smooth: a reduced group scheme over an algebraically closed field is smooth (1.5.3 in Conrad's notes).

Connected: by def. ■

Now we start to make use of $\mathcal{L}$. The restriction $\mathcal{L}_Y$ of L to Y is ample and $T_y^*(\mathcal{L}_Y) \cong \mathcal{L}_Y$ for all $y \in Y(k)$.

Claim: $\mathcal{L}_Y \otimes [-1]^*\mathcal{L}_Y$ is both ample and trivial on Y.

Proof of the Claim: Ampleness: $\mathcal{L}_Y$ is ample and $[-1]$ is an automorphism on Y.

Trivialnesss: Similar to the proof of the Seesaw Theorem

$$m^*\mathcal{L}_Y \otimes p_1^*\mathcal{L}_Y^{-1} \otimes p_2^*\mathcal{L}_Y^{-1}|_{\{y\}\times Y} \cong T_y^*\mathcal{L}_Y \otimes \mathcal{O}_Y \otimes \mathcal{L}_Y^{-1} \cong \mathcal{O}_Y.$$

Thus this line bundle is trivial on $\{y\} \times Y$ for every $y \in Y(k)$. By Seesaw it is of the form $p_2^*\mathcal{M}$ where $\mathcal{M}$ is a line bundle over Y. One can construct a section for p_2 and show that $\mathcal{M}$ must be trivial.

Now consider the morphism $(\mathrm{id}, -\mathrm{id}) : Y \to Y \times Y$ defined by $y \mapsto (y, -y)$, by observing k-valued points

$$(\mathrm{id}, -\mathrm{id})^*(m^*\mathcal{L}_Y \otimes p_1^*\mathcal{L}_Y^{-1} \otimes p_2^*\mathcal{L}_Y^{-1}) \cong \mathcal{O}_X \otimes \mathcal{L}_Y^{-1} \otimes [-1]^*\mathcal{L}_Y^{-1}.$$

■

The claim above $\implies$ Y is quasi-affine $\implies$ $Y = \{*\}$ $\implies$ $K(L)$ is finite (otherwise we can move a curve to 0).

(ii) $\implies$ *(i)*: Since $H(D) \subset K(\mathcal{L})$.

(i) $\implies$ *(iii)*: $\underline{\mathcal{L}^{\otimes 2} \text{ is globally generated}}$: Since X is quasi-compact and $L^{\otimes 2}$ is of finite type,

$\mathcal{L}^{\otimes 2}$ is finitely globally generated $\iff$ $\mathcal{L}^{\otimes 2}$ is globally generated $\iff$ for any point $u \in X(k)$, there exists a global section $s \in \mathcal{L}^{\otimes 2}$ such that s_u does not vanish at u. (some people like to call it base-point free)

So fix a point u, there exists an $x \in X$ such that $u \pm x \notin$ Supp D (otherwise for all $x \in X(k)$, $x \in \pm$ (Supp D $-u$) and this means that $u \notin$ Supp $(T_x^*(D) + T_{-x}^*(D))$.

By Thm of the Square,

$$\mathcal{L}^{\otimes 2} \cong \mathcal{O}_X(2D) \cong \mathcal{O}_X(T_x^*(D) + T_{-x}^*(D)).$$

Recall that we can take $1 \in \Gamma(X, \mathcal{O}_X(T_x^*(D) + T_{-x}^*(D))$ which does not vanish at (Supp $(T_x^*(D) + T_{-x}^*(D)))^c$. So it does not vanish at u! Note that it suffices to check that their exists a global section does not vanish at closed points, because if some global section vanishes at gen points it vanishes at all closed points above it!

Remark. *In the above argument, we did not use the fact that $H(D)$ is finite. Thus over X, every line bundle of the form $\mathcal{L}^{\otimes 2}$ where $\mathcal{L}$ is associated to an effective divisor is globally generated.*

$\underline{\varphi \text{ is finite}}$: φ is finite $\iff$ φ is quasi-finite (by properness) and quasi-finiteness suffices to be checked on closed points. One can refer to: https://math.stackexchange.com/questions/146186/finite-fibers-over-closed-points. Suppose φ is not quasi-finite, then there exists an irreducible curve C such that $\varphi(C) =$\{one point\} (consider the fiber product between X and some point and take a dimension 1 irreducible closed subset).

Claim: There exists $x_0 \in X(k)$ such that $(D + x_0) \bigcap C = \varnothing$.

Proof of the Claim: Recall that divisors that are linearly equivalent to $D + D$ are exactly those of the form $\varphi^{-1}(H)$ where H is a hyperplane in $\mathbb{P}^N$. One can refer to: https://math.stackexchange.com/questions/356710/the-fibers-of-a-map-to-mathbbp1. Thus one either have $C \subset T_x^*(D) + T_{-x}^*(D)$ or $C \bigcap T_x^*(D) + T_{-x}^*(D) = \varnothing$. Fix some $c_0 \in C(k)$, by the arguments above there exists some x_0 such that $c_0 \notin \text{Supp }(T_{x_0}^*(D) + T_{-x_0}^*(D))$ so at least C and $D + x_0$ are disjoint. ■

Now we state a general fact

Lemma 1.4. *If C is a curve on X and E is an irreducible divisor on X such that $C \bigcap E = \emptyset$, then E is invariant under translation by $x_1 - x_2$ where $x_1, x_2 \in C$.*

Sketch of the proof of the lemma: $\mathcal{N} = \mathcal{O}_X(E)$ is trivial on C. Thus $T_x^*\mathcal{N}|_C$ has degree 0 for all $x \in X(k)$. As intersecting at finitely many points makes the degree positive, $T_x(C)$ and E are disjoint or $T_x(C) \subset E$. Let $x_1,\ x_2 \in C(k)$ and $y \in E(k)$. Then $T_{x_2-y}^*(C)(k)$ and $E(k)$ meet at y. So $T_{x_2-y}^*(C)(k) \subset E(k)$, in particular $T_{x_2-y}^*(x_1) \in E(k)$ and $y - x_2 + x_1 \in E(k)$.

Write $D = \sum n_i D_i$ with D_i irreducible, then by Claim+Lemma each $D_i + x_0$ is invariant under translation by all points $x_1 - x_2$, $x_i \in C$. So is $D + x_0$ and D, which contradicts to the finiteness of (i). □

Corollary 1.5. *Let X be an abelian variety. Then X is projective.*

Proof. Reduced to: construct an ample line bundle $\mathcal{L}$ over X.

Let $\operatorname{Spec} A$ be an affine open neighbourhood of 0.

Claim: Suppose X is Noetherian, normal and separated, $\operatorname{Spec} A \subset X$ then $D := X - \operatorname{Spec} A$ is of pure codimension 1.

Proof of the Claim: (one can draw a picture) Suppose $\eta \in D$ with $\operatorname{ht}(\eta) \geq 2$. Then choose an affine neighbourhood V of η such that gen points of $D \bigcap V = \{\eta\}$.

Then argue that V and $\operatorname{Spec} A \bigcap V$ has the same set of codimension 1 points. Use the algebraic Hartog's lemma (requires Noetherianess and normality of X) to see that they have the isomorphic global section, and both are affine). ■

As D is a closed subset of a Noetherian scheme there are only finitely many irreducible components, D can also be expressed as a Weil divisor. Thus we can define $H(D)$.

$H(D)$ stabilizes D $\iff$ $H(D)$ stabilizes $\operatorname{Spec} A$ $\implies$ $H(D) \subset \operatorname{Spec} A$ (because $0 \in \operatorname{Spec} A$). $H(D)$ is proper and lies in an affine open neighbourhood means H is finite. By the theorem above, $\mathcal{L}$ is ample. □

Remark. *In the proof we constructed an ample line bundle, but we can even construct an ample and symmetric line bundle by taking $L \otimes [-1]^*L$.*

2 Structure of n-torsion subgroups of $X(k)$

Definition 2.1. *Define $X[n]$ to be the pullback of the following diagram*

$$\begin{tikzcd} X[n] \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\ X \arrow[r] & X \end{tikzcd}$$

Warning: $X[n]$ might not be reduced if char $k \mid n$.

Motivation: when $k = \mathbb{C}$, $X(\mathbb{C})$ is a complex torus (refer to Chapter 1 (2) in Mumford's book), i.e.

$$X(\mathbb{C}) \cong \mathbb{C}^g/\Lambda$$

where Λ is a full lattice. In this case it is easy to see that $X[n](k) \cong \frac{1}{n}\Lambda/\Lambda \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$. When $g = 1$ one can even draw a picture.

One can see that $X[n](k)$ is the n-torsion subgroup of $X(k)$. To study the n-torsion subgroup, it is natural to study the homomorphism $[n]$ first.

Definition 2.2 (Degree of a morphism)**.** *Let X and Y be varieties both of dimension n and let $f : X \to Y$ be a surjective morphism. The function fields $k(X)$ and $k(Y)$ are of transcendental degree n so via f^*, $k(X)$ is a finite extension of $k(Y)$. Then we define the degree d of f to be the degree of the extension $[k(X) : k(Y)]$.*

Now we state some properties of $[n]$.

Proposition 2.3. *i) For all $n \geq 1$, $[n]$ is a finite flat surjection. Thus $X(k)$ is a divisible group.*
ii) $\deg[n] = n^{2g}$ *where* $g = \dim X$.

Proof. Claim: $X[n](k)$ is finite.

The claim above $\implies$ $[n]$ is quasi-finite (again only checked on closed points) because $[n]^{-1}(x)$ is either empty or $x_0 + X[n](k)$ for some $x_0 \in X(k) \implies [n]$ is finite.

By Going-up theorem finite morphisms does not contract any chain so the image of a length n chain still has length n and we can reach the generic point. Thus $[n]$ is surjective.

Proof of the Claim: Let $\mathcal{L}$ be an ample line bundle on X (we proved it exists just now!). Then by Corollary 3 in the last talk

$$[n]^*\mathcal{L} \cong \mathcal{L}^{\frac{n(n+1)}{2}} \otimes [-1]^*\mathcal{L}^{\frac{n(n-1)}{2}}.$$

Again we can see that $[n]^*\mathcal{L}$ is ample but $[n]^*\mathcal{L}$ cannot be trivial on any positive dimensional subvariety. Because $[n]^*\mathcal{L}$ is trivial on $X[n]$ (think about pulling back through $\operatorname{Spec} k$ in the Cartesian diagram), $X[n](k)$ must be finite. ■

Flatness of $[n]$: refer to the Miracle Flatness Theorem in Matsumura Ch 23.1.

$\deg[n] = n^{2g}$: For any line bundle we have the following formula:

$$\deg(\mathcal{L}^n) = n^g \deg \mathcal{L}.$$

where $g = \dim X$. Let $\mathcal{L}$ be an ample line bundle over X. Replace $\mathcal{L}$ by $\mathcal{L} \otimes [-1]^*\mathcal{L}$ which is a symmetric ample line bundle. Then $\deg([n]^*\mathcal{L}) = \deg(\mathcal{L}^{n^2}) = n^{2g} \deg \mathcal{L}$. (here use Corollary 3 again)

Finally conclude use the fact: under pullback by a finite flat morphism and if the target is proper and integral, the degree is multiplied by the degree of the morphism (Stacks project 0BEX/Conrad Prop 4.1.11). □

Next: study the structure of $X[n](k)$ as a group. Let $p = \operatorname{char} k$.

Fact: Let d' be the separable degree of the extension, then the cardinality of $f^{-1}(y)$ is d' for y in a dense open subset of Y. Thus the n-torsion group can be studied by considering the separable degree of $[n]$.

Proposition 2.4. *If $p \nmid n$, $X[n](k) \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$.*

Proof. $[n]^*$ induces a separable extension on $[k(X) : k(Y)]$ since $p \nmid n$, so $|X[n](k)| = n^{2g}$ and we know the cardinality of all m-torsion group in $X[n](k)$ is m^{2g} where $m \mid n$. By elementary group theory we are done. □

Proposition 2.5. *$X_{p^m} \cong (\mathbb{Z}/p^m\mathbb{Z})^i$ for some $0 \leq i \leq g$.*

Proof. (Sketched)

Reduce to show $X[p](k) = (\mathbb{Z}/p\mathbb{Z})^i$ because $X[n]$ is divisible.

Reduce to: show $[p]^* : k(X) \to k(X)$ factors through $k(X)^p$. This implies that the inseparable degree is at least p^g.

The universal differential $d : k(X) \to \Omega^1_{k(X)/k}$ has kernel $k(X)^p$.

$d([p]^*f) = [p]^*df = 0$ for all $f \in k(X)$. Thus it suffices to show that $[p]^*$ kills $\Omega^1_{k(X)/k}$.

From the discussion in Ch 4, $d[p]$ maps $T_{X,0}$ to 0. Since $[p]$ commutes with translation, $[p]^*$ is everywhere 0 on $\Omega^1_{X/k}$. Looking at the generic point $[p]^*$ kills $\Omega^1_{k(X)/k}$. □

Remark. *Indeed we classified all $X[n]$ since by Chinese Remainder Thm we can split to the torsion subgroups for different p_i's.*